Linear Algebra I

02/02/2024, Friday, 8:30 – 10:30

You are **NOT** allowed to use any type of calculators.

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1 Determinants $\qquad 10 + 10 = 20$ pts

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Consider the matrix

$$M = \begin{bmatrix} I_n & \boldsymbol{x} \\ \boldsymbol{y}^T & 1 \end{bmatrix}.$$

(a) Show that

$$\det(M) = \det(I_n - \boldsymbol{x}\boldsymbol{y}^T) = 1 - \boldsymbol{y}^T\boldsymbol{x}.$$

(b) Assume that $\boldsymbol{y}^T\boldsymbol{x} \neq 1$ and find the inverse of M.

2 Eigenvalues/vectors and diagonalization $\qquad 5 + 5 + 5 + 10 = 25$ pts

Consider the matrix

$$M = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

(a) Show that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of M. Find the corresponding eigenvalue, say λ_1.

(b) Show that $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq 0$ is an eigenvector of M if

$$a + b + c = 0.$$

Find the corresponding eigenvalue, say λ_2.

(c) Does M have an eigenvalue other than λ_1 and λ_2?

(d) Find an orthogonal diagonalizer for M.

3 Subspaces $10 + 10 = 20$ pts

Consider the sets

$$S_1 = \{\boldsymbol{x} \in \mathbb{R}^2 \mid \boldsymbol{x}^T\boldsymbol{x} = a\} \quad \text{and} \quad S_2 = \left\{\boldsymbol{x} \in \mathbb{R}^2 \mid \boldsymbol{x}^T \begin{bmatrix} a & a \\ a & a \end{bmatrix} \boldsymbol{x} = 0\right\}.$$

Determine all values of $a \in \mathbb{R}$ such that

(a) S_1 is a subspace.

(b) S_2 is a subspace.

4 Jordan canonical form $3 + 10 + 12 = 25$ pts

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

(b) Is A diagonalizable? Why?

(c) Find the Jordan canonical form of A.

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10 pts free